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Pointwise and Renyi Dimensions of an Invariant Measure of Random Dynamical Systems with Jumps

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We estimate the lower pointwise dimension and the generalized Renyi dimension of an invariant measure of random dynamical systems with jumps. It is worthwhile to note that the dimensions are a useful tool in studying the Hausdorff dimension of measures and sets. Our model generalizes Markov processes corresponding to iterated function systems and Poisson driven stochastic differential equations. It can be used as a description of many physical and biological phenomena.

KEY WORDS: Dynamical systems, Markov semigroup, invariant measure, pointwise and Renyi dimension.
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1. INTRODUCTION

Let $(Y, \|\cdot\|)$ be a separable Banach space, $\mathbb{R}_+ = [0, +\infty)$ and $I = \{1, \ldots, N\}$, $S = \{1, \ldots, K\}$. We first define our system.

Let $\Pi_i : \mathbb{R} \times Y \to Y, i \in I$, be a finite sequence of dynamical systems; $p_i : Y \to [0, 1],\ \ i \in I, \overline{p}_s : Y \to [0, 1],\ \ s \in S$ be probability vectors and $[p_{ij}]_{i,j\in I}$, $p_{ij} : Y \to [0, 1],\ \ i, j \in I$, be a matrix of probabilities.

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $\{t_n\}_{n\geq 0}$ be a sequence of random variables $t_n : \Omega \to \mathbb{R}_+$ with $t_0 = 0$ and such that the increments $\Delta t_n = t_n - t_{n-1}$, $n \in \mathbb{N}$, are independent and have the same density $g(t) = \lambda e^{-\lambda t}$.

Finally, let $q_s : Y \to Y, s \in S$ be a family of continuous functions. In the sequel we denote the system by (Π, q, p).

The action of randomly chosen dynamical systems, with randomly chosen jumps, at random moments $\{t_n\}_{n\geq 0}$ corresponding to the system (Π, q, p) can be roughly described as follows.

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We choose an initial point $x \in Y$ and randomly select a transformation Π_i from a set $\{\Pi_1, \dots, \Pi_N\}$ in such a way that probability of choosing Π_i is equal to $p_i(x)$ and define

$$X(t) = \Pi_i(t, x) \quad \text{for} \quad 0 \le t < t_1.$$

Next at the random moment t_1, at the point $\Pi_i(t_1, x)$ we choose a jump q_s from a set $\{q_1, \dots, q_K\}$ with probability $\overline{p}_s(\Pi_i(t_1, x))$. Then we define

$$x_1 = q_s(\Pi_i(t_1, x)).$$

After that we choose Π_{i_1} with probability $p_{ii_1}(x_1)$, define

$$X(t) = \Pi_{i_1}(t - t_1, x_1) \quad \text{for} \quad t_1 < t < t_2$$

and at the point $\Pi_{i_1}(t_2 - t_1, x_1)$ we choose q_{s_1} with probability $\overline{p}_{s_1}(\Pi_{i_1}(t_2 - t_1, x_1))$. Then we define

$$x_2 = q_{s_1}(\Pi_{i_1}(t_2 - t_1, x_1))$$

and so on.

Finally, given x_n, $n \ge 2$ we choose Π_{i_n} in such a way that the probability of choosing Π_{i_n} is equal to $p_{i_{n-1}i_n}(x_n)$, define

$$X(t) = \Pi_{i_n}(t - t_n, x_n) \quad \text{for} \quad t_n < t < t_{n+1}$$

and at the point $\Pi_{i_n}(\Delta t_{n+1}, x_n)$ we choose q_{s_n} with probability $\overline{p}_{s_n}(\Pi_{i_n}(\Delta t_{n+1}, x_n))$. Then we define

$$x_{n+1} = q_{s_n}(\Pi_{i_n}(\Delta t_{n+1}, x_n)).$$

It is well know (see ref. 5) that the random dynamical system with jumps generates a semigroup of Markov operators $\{P^t\}_{t\ge0}$ acting on the space of Borel measures on Y. It is also known (see refs. 7 and 9) that under suitable conditions there exists a Markov operator P such that $X(t_k)$ has the distribution $P^k\mu$ if μ is the distribution of x_0. Relations between the measure μ_0 invariant with respect to the Markov operator P and the measure μ_* invariant with respect to the Markov semigroup $\{P^t\}_{t\ge0}$ were studied in refs. 6 and 15.

In our consideration we only assume that Y is a separable Banach space. The model under consideration is a particular case of so-called piecewise-deterministic Markov processes introduced by Davis.(1) The method of proving the existence of an invariant measure used by Davis is not well adapted to the infinite-dimensional case. The main difficulty is to show that piecewise-deterministic Markov processes satisfy some ergodic properties on compact sets. However assumption of compactness is restrictive if we want to apply our model in physics and biology.

The system considered in this paper generalizes some very important and widely studied cases, namely dynamical systems generated by learning systems, Poisson driven stochastic differential equations, iterated function system with an infinite family of transformations and random evolutions. A large class of applications of such models, both in physics and biology, is worth mentioning here : the short noise, the photoconductive detectors, the growth of the size of structural population, the motion of relativistic particles, both fermions and bosons, and many others (see refs. 3, 10, and 12). On the other hand, it should be noted that most Markov chains, appear among other things, in statistical physics, and may be represented as iterated function systems (see ref. 11). Recently, iterated function systems have been used in studing invariant measures for the Ważewska partial differential equation which describes the process of reproduction of the red blood cells.[13,14] Similar nonlinear first-order partial differential equations frequently appear in hydrodynamics.[19]

The dimension of invariant sets is among the most important characteristics of dynamical systems. Hausdorff dimension, introduced in 1919, is a notion of size usefull for distinguishing between sets of Lebesgue measure zero. The notion was widely investigated and widely used, among other in the theory of dynamical systems, where many interesting invariant sets are null in the sense of Lebesgue. Unfortunately, in many cases the straightforward calculation of the Hausdorff dimension was very difficult. This prompted researchers to introduce other characteristics. Among them are capacity dimension, pointwise dimension, correlation dimension, Renyi dimension, etc.

We estimate the lower pointwise dimension and the generalized Renyi dimension of the invariant measure of the random dynamical system with jumps. It is worthwhile to note that the dimensions are a useful tools in studing the Hausdorff dimension of measures and sets.[13,18]

The result of this paper is related to the papers.[13,17,20] In Szarek[20] considered the lower pointwise dimension of invariant measures related to Poisson driven stochastic differential equations.

Using the notion of the Levy concentration function Lasota and Myjak[13] introduced the concentration dimension (the generalized Renyi dimension) and using this dimension they calculated some bounds of the concentration dimension of fractals and semifractals generated by iterated function systems.

2. PRELIMINARIES

Let $(\mathfrak{X}, \varrho)$ be a complete separable metric space. By $B(x, r)$ we denote the open ball with center at x and radius r. For a subset A of $\mathfrak{X}$, $cl A$, $diam\ A$, and 1_A stands for the closure of A, diameter of A and the characteristic function of A, respectively.

By $\mathcal{B}(\mathfrak{X})$ we denote the σ-algebra of Borel subsets of $\mathfrak{X}$ and by $\mathcal{M} = \mathcal{M}(\mathfrak{X})$ the family of all finite Borel measures on $\mathfrak{X}$. By $\mathcal{M}_1 = \mathcal{M}_1(\mathfrak{X})$ we denote the space of all $\mu \in \mathcal{M}$ such that $\mu(\mathfrak{X}) = 1$ and by $\mathcal{M}_s$ the space of all finite signed Borel measures on $\mathfrak{X}$. The elements of $\mathcal{M}_1$ are called *distributions*.

As usual, by $B(\mathfrak{X})$ we denote the space of all bounded Borel measurable functions $f : \mathfrak{X} \to \mathbb{R}$ and by $C(\mathfrak{X})$ the subspace of all continuous functions. Both spaces are considered with the supremum norm $\|\cdot\|_0$.

For $f \in B(\mathfrak{X})$ and $\mu \in \mathcal{M}_s$ we write

$$\langle f, \mu\rangle = \int_{\mathfrak{X}} f(x)\mu(dx).$$

An operator $P : \mathcal{M} \to \mathcal{M}$ is called a *Markov* operator if

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad \text{for} \quad \lambda_1, \lambda_2 \in \mathbb{R}_+ \quad \text{and} \quad \mu_1, \mu_2 \in \mathcal{M}$$

and

$$P\mu(\mathfrak{X}) = \mu(\mathfrak{X}) \quad \text{for} \quad \mu \in \mathcal{M}.$$

A linear operator $U : B(\mathfrak{X}) \to B(\mathfrak{X})$ is called *dual* to P if

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for} \quad f \in B(\mathfrak{X}) \quad \text{and} \quad \mu \in \mathcal{M}.$$

A measure $\mu_0 \in \mathcal{M}_1$ is called *invariant* or *stationary* with respect to a Markov operator P if $P\mu_0 = \mu_0$.

A family of Markov operators $\{P^t\}_{t\geq 0}$ is called a *semigroup* if $P^{t+s} = P^t P^s$ for all $t, s \in \mathbb{R}_+$ and P^0 is the identity operator on $\mathcal{M}$. A measure μ_* is called *invariant* with respect to P^t if $P^t\mu_* = \mu_*$ for every $t \geq 0$.

Now introduce the class Φ of functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) φ is continuous and $\varphi(0) = 0$;
(ii) φ is nondecreasing and concave, i.e.

$$\sum_{k=1}^{n} \alpha_k \varphi(t_k) \leq \varphi\left(\sum_{k=1}^{n} \alpha_k t_k\right), \quad \text{where} \quad \alpha_k \geq 0, \quad \sum_{k=1}^{n} \alpha_k = 1;$$

(iii) $\varphi(x) > 0$ for $x > 0$ and $\lim_{x\to\infty} \varphi(x) = \infty$.

By Φ_0 we denote the family of all functions satisfying conditions (i) and (ii). Observe that for every $\varphi \in \Phi$ the function $\rho_\varphi = \varphi \circ \rho$ is again a metric on $\mathfrak{X}$. Moreover ρ_φ is equivalent to ρ.

As usual, for $A \subset \mathfrak{X}$, $s > 0$ and $\delta > 0$ we define

$$\mathcal{H}^s_\delta(A) = \inf \sum_{i=1}^{\infty} (diam\ E_i)^s$$

where the infinium is taken over all countable covers $\{E_i\}$ of A such that $diam E_i < \delta$. Then

$$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A)$$

defines the Hausdorff's-dimensional measure. The Hausdorff dimension of A is defined by the formula

$$dim_H A = \sup\{s > 0 : \mathcal{H}^s(A) > 0\}.$$

Here we admit that sup = 0.

Let $\mu \in \mathcal{M}_1$. The Hausdorff dimension of μ is defined by the formula

$$dim_H \mu = \inf\{\dim_H A : A \in \mathcal{B}(\mathfrak{X}) \quad \text{and} \quad \mu(A) = 1\}.$$

Let $\mu \in \mathcal{M}$ and $x \in \mathfrak{X}$. We define the lower pointwise dimension of μ at x by the formula

$$\underline{d}\mu(x) = \liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},$$

(where we assume that $\log 0 = -\infty$.)

Given $\mu \in \mathcal{M}$ we define the Levy concentration function $Q_\mu : (0, +\infty) \to \mathbb{R}_+$ by the formula (see ref. 16)

$$Q_\mu(r) = \sup\{\mu(B(x, r), x \in \mathfrak{X}\} \quad \text{for} \quad r > 0.$$

Further for a measure $\mu \in \mathcal{M}_1$ we define the lower and the upper concentration dimension of μ by the formulae

$$\underline{dim}_L \mu = \liminf_{r \to 0} \frac{\log Q_\mu(r)}{\log r}$$

and

$$\overline{dim}_L \mu = \limsup_{r \to 0} \frac{\log Q_\mu(r)}{\log r}.$$

If $\underline{dim}_L \mu = \overline{dim}_L \mu$ then this common value is called the generalized Renyi dimension of μ and it is denoted by $dim_L \mu$. (In many papers $dim_L \mu$ is called concentration dimension of μ[13,15]).

3. INVARIANT MEASURES

Assume that we are given the system (Π, q, p) on a separable Banach space defined in Sec. 1. Recall that $\Pi_k : \mathbb{R} \times Y \to Y, k \in I$, is a dynamical system, i.e.

(i)

$$\Pi_k(0, x) = x \quad \text{for every} \quad k \in I, \quad x \in Y$$

and

(ii)

$$\Pi_k(s + t, x) = \Pi_k(s, (\Pi_k(t, x)) \quad \text{for every} \quad s, t \in \mathbb{R}, \quad k \in I$$

and x∈ Y.

We assume that $\Pi_k : \mathbb{R} \times Y \to Y$, $k \in I$ and $q_s : Y \to Y$, $s \in S$ are continuous and that there exists $x_* \in Y$ such that

$$\int_{\mathbb{R}_+} e^{-\lambda t} \|q_s(\Pi_j(t, x_*)) - q_s(x_*)\| \, dt < \infty \quad \text{for} \quad j \in I, \quad s \in S. \tag{3.1}$$

Moreover suppose we are given probability vectors $(p_1, \dots, p_N)$, $(\overline{p}_1, \dots, \overline{p}_K)$ where $p_i : Y \to [0, 1]$, $\overline{p}_s : Y \to [0, 1]$ and a probability matrix $[p_{ij}]_{i,j \in I}$ with $p_{ij} : Y \to [0, 1]$.

We assume that functions $\overline{p}_s$, $s \in S$ and p_{ij}, $i, j \in I$, satisfy the Dini condition, i.e.

$$\sum_{j=1}^{N} |p_{ij}(x) - p_{ij}(y)| \le \omega_1(\|x - y\|) \quad \text{for} \quad x, y \in Y, \ i \in I,$$

$$\sum_{s \in S} |\overline{p}_s(x) - \overline{p}_s(y)| \le \omega_2(\|x - y\|) \quad \text{for} \quad x, y \in Y \tag{3.2}$$

where the functions $\omega_1, \omega_2 \in \Phi_0$ and satisfy the Dini condition

$$\int_0^{\varepsilon} \frac{\omega_i(t)}{t} dt < \infty \quad \text{for some} \quad \varepsilon > 0.$$

Moreover, we assume that there exist constants $L \ge 1$ and $\alpha \in \mathbb{R}$ such that

$$\sum_{j=1}^{N} p_{ij}(y) \|\Pi_j(t, x) - \Pi_j(t, y)\| \le L e^{\alpha t} \|x - y\| \quad \text{for} \quad x, y \in Y, \ \ i \in I, \ \ t \ge 0. \tag{3.3}$$

Finally we assume that there exists a constant $L_q > 0$ such that

$$\sum_{s \in S} \overline{p}_s(x) \|q_s(x) - q_s(y)\| \le L_q \|x - y\| \quad \text{for} \quad x, y \in Y. \tag{3.4}$$

Let $\{t_n\}_{n \ge 0}$ be the sequence of random variables introduced in Sec. 1.

We consider sequences of random variables : $\{x_n\}_{n \ge 0}$, $x_n : \Omega \to Y$, $\{\xi_n\}_{n \ge 0}$, $\xi_n : \Omega \to I$, $\{\eta_n\}_{n \ge 1}$, $\eta_n : \Omega \to S$, auxiliary random variables $\{y_n\}_{n \ge 1}$, $y_n : \Omega \to Y$ and stochastic process $\{X(t)\}_{t \ge 0}$, $X(t) : \Omega \to Y$. We assume that they are related by

$$y_n = \Pi_{\xi_{n-1}}(t_n - t_{n-1}, x_{n-1}), \quad x_n = q_{\eta_n}(y_n) \quad \text{for} \quad n \ge 1$$

$$\mathbb{P}(\xi_0 = k | x_0 = x) = p_k(x),$$

$$\mathbb{P}(\xi_n = i | x_n = x \quad \text{and} \quad \xi_{n-1} = k) = p_{ki}(x),$$

$$\mathbb{P}(\eta_n = s | y_n = y) = \overline{p}_s(y) \quad \text{for} \quad n \ge 1, \ x, y \in Y, \ k, i \in I \quad \text{and} \quad s \in S$$

and

$$X(t) = \Pi_{\xi_{n-1}}(t - t_{n-1}, x_{n-1}) \quad \text{for} \quad t_{n-1} < t < t_n,$$
$$X(t_n) = x_n. \tag{3.5}$$

Assume that $\{\xi_n\}_{n\geq 0}$ and $\{\eta_n\}_{n\geq 0}$ are independent upon $\{t_n\}_{n\geq 0}$ and that for every $n \in \mathbb{N}$ the variables $\eta_1, \dots, \eta_{n-1}, \xi_1, \dots, \xi_{n-1}$ are also independent.

Simple consideration shows that the process $\{X(t)\}_{t\geq 0}$ is not Markovian. In order to use the machinery of Markov operators we must to remodel the process $\{X(t)\}_{t\geq 0}$ in such a way that the new process becomes Markovian.

In this purpose consider the space $Y \times I$ endowed with the metric ϱ given by

$$\varrho\big((x, i), (y, j)\big) = \|x - y\| + \varrho_0(i, j) \quad \text{for} \quad x, y \in Y, \quad i, j \in I, \tag{3.6}$$

where

$$\varrho_0(i, j) = \begin{cases} c, & \text{if } i \neq j, \\ 0, & \text{if } i = j \end{cases}$$

with the constant c suitably choosen.

Now we define a stochastic proces $\{\xi(t)\}_{t\geq 0}, \xi(t) : \Omega \to I$ by

$$\xi(t) = \xi_n \quad \text{for} \quad t_n < t < t_{n+1}, \quad n = 0, 1, \dots$$

and we consider a stochastic process $\{(X(t), \xi(t))\}_{t\geq 0}$, $(X(t), \xi(t)) : \Omega \to Y \times I$. It is easy to check that this process admits the Markov property.

This process generates a semigroup $\{T^t\}_{t\geq 0}$ defined by

$$T^t f(x, i) = E(f((X(t), \xi(t))_{(x,i)})) \quad \text{for} \quad f \in C(Y \times I), \tag{3.7}$$

where $E(f(X(t), \xi(t))_{(x,i)})$ denotes the mean value of $f((X(t), \xi(t))_{(x,i)})$. Now we define semigroup operators $\{P^t\}_{t\geq 0}$, $P^t : \mathcal{M}_1(Y \times I) \to \mathcal{M}_1(Y \times I)$ by

$$\langle P^t \mu, f\rangle = \langle \mu, T^t f\rangle \quad \text{for} \quad f \in C(Y \times I), \mu \in \mathcal{M}_1(Y \times I). \tag{3.8}$$

In many applications we are mostly interested in the values of the process $X(t)$ at the switching points t_n. Now we consider a sequence of random variables $\{(x_n, \xi_n)\}_{n\geq 0}$. Clearly $\big(x_n, \xi_n\big) : \Omega \to Y \times I$ and admits the Markov property.

Let μ_0 be a distribution of the initial random variable (x_0, ξ_0), i.e.

$$\mu_0(A) = \mathbb{P}\big((x_0, \xi_0) \in A\big) \quad \text{for} \quad A \in \mathcal{B}(Y \times I).$$

For $n \in \mathbb{N}$ we denote by μ_n the distribution of (x_n, ξ_n), i.e.

$$\mu_n(A) = \mathbb{P}\big((x_n, \xi_n) \in A\big) \quad \text{for} \quad A \in \mathcal{B}(Y \times I).$$

Then there exists (see ref. 7) a Markov–Feller operator $P : \mathcal{M}(Y \times I) \to \mathcal{M}(Y \times I)$ such that

$$\mu_{n+1} = P\mu_n \quad \text{for every} \quad n \in \mathbb{N}.$$

Moreover, the operator P is given by the formula

$$P\mu(A) = \sum_{j\in I} \int_{Y\times I} \int_0^{+\infty} \sum_{s\in S} \lambda e^{-\lambda t} \cdot 1_A(q_s(\Pi_j(t,x)), j) p_{ij}(x) \times \overline{p}_s(\Pi_j(t,x))\, dt\, \mu(dx\, di) \tag{3.9}$$

and its dual operator U by the formula

$$Uf(x,i) = \sum_{j\in I} \int_0^{+\infty} \sum_{s\in S} f(q_s(\Pi_j(t,x)), j) p_{ij}(x) \overline{p}_s(\Pi_j(t,x)) \lambda e^{-\lambda t}\, dt.$$

Theorem 3.1. *Assume that the system* (Π, p, q) *satisfies the conditions (3.1)–(3.4). Assume, moreover that*

$$LL_q + \frac{\alpha}{\lambda} < 1. \tag{3.10}$$

Then the operator P *defined by (3.9) admits an invariant measure.*

Proof. The proof can be found in ref. 7. □

It was shown in ref. 6 that there is a one-to-one mapping between the invariant probability measure for $(X(t), \xi(t))$ and the invariant probability measure for the Markov chain given by post jump locations (x_n, ξ_n). However, the proof is given for $\mathbb{R}^n$ but it remains valid for Banach spaces.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. then the semigroup* $\{P^t\}_{t\geq 0}$ *given by (3.8) admits an invariant measure* μ_**. Moreover, if* $\mu_0 \in \mathcal{M}_1(Y \times I)$ *is the invariant measure for Markov operator* P *then*

$$\mu_* = G\mu_0$$

where

$$G\mu(A) = \sum_{i\in I} \int_0^{+\infty} \int_{Y\times I} 1_A(\Pi_i(t,x)), i) p_{ki}(x) \lambda e^{-\lambda t}\, dt\, d\mu(x,k)$$
$$for \quad A \in \mathcal{B}(Y \times I) \quad \mu \in \mathcal{M}.$$

Proof. The proof follows from Theorem 3.1[6] and Theorem 3.1. □

4. LOWER POINTWISE DIMENSION OF AN INVARIANT MEASURE

We start with the following technical observation.

Lemma 4.1. *Assume that μ_* is the invariant measure of the semigroup $\{P^t\}_{t\geq 0}$ given by (3.8). Then*

$$\mu_*(A) \geq e^{-\lambda t} \int_{Y\times I} 1_A(\Pi_i(t,x), i) d\mu_*(x,i) \quad for \quad A \in \mathcal{B}(Y \times I). \tag{4.1}$$

Proof. Fix $A \in \mathcal{B}(Y \times I)$ and $t \geq 0$. We have

$$\mu_*(A) = P^t \mu_*(A) =< U^t 1_A, \mu_* >= \int_{Y\times I} E 1_A((X(t), \xi(t))_{(x,i)}) d\mu_*(x,i). \tag{4.2}$$

Fix $(x,i) \in Y \times I$ and observe that

$$\begin{aligned} E 1_A((X(t), \xi(t))_{(x,i)}) &= \int_{\Omega} 1_A((X(t), \xi(t))_{(x,i)}(\omega)) \mathbb{P}(d\omega) \\ &\geq \int_{\Omega_0(t)} 1_A((X(t), \xi(t))_{(x,i)}(\omega)) \mathbb{P}(d\omega), \end{aligned}$$

where $\Omega_0(t) = \{\omega \in \Omega : t \leq t_1(\omega)\}$.

Since $((X(t), \xi(t))_{(x,i)}(\omega)) = (\Pi_i(t,x), i)$ for $\omega \in \Omega_0(t)$ and $\mathbb{P}(\Omega_0(t)) = e^{-\lambda t}$ we obtain

$$E 1_A(X(t), \xi(t))_{(x,i)}) \geq e^{-\lambda t} 1_A(\Pi_i(t,x), i).$$

From (4.2) the statement of Lemma 4.1 follows immediately. □

Theorem 4.1. *Let the assumptions of Theorem 3.2 hold and let μ_* be the unique invariant measure with respect to the semigroup $\{P^t\}_{t\geq 0}$ given by (3.8). Assume that*

$$\sigma = \inf_{x\in Y, i,j \in I} p_{ij}(x) > 0. \tag{4.3}$$

Moreover, assume that for every $j \in I$ there exists a constant $l_j \in (0,1]$ such that

$$\|\Pi_j(t,x) - \Pi_j(t,y)\| \geq l_j \|x - y\| \quad for \quad x, y \in Y,\ t \geq 0 \tag{4.4}$$

and for every $x \in Y$ and $j \in I$ there exist $\delta_j > 0$ and $c_{x,j} > 0$ such that

$$\|\Pi_j(t,x) - x\| \geq c_{x,j} t \quad for \quad 0 < t < \delta_j. \tag{4.5}$$

Then

$$\underline{d}\mu_*(x,i) \geq \frac{\log 3}{\log 3 + \log \frac{L}{\sigma \min_j l_j}} \quad for \quad (x,i) \in Y \times I,$$

where L is the constant appearing in the condition (3.3).

Proof. We consider two cases : $\alpha \geq 0$ and $\alpha < 0$. □

Case I. Suppose first that $\alpha \geq 0$. Let $\overline{x} \in Y$ and $k \in I$ be fixed. Choose $\varepsilon > 0$ such that

$$r_0 = \frac{\ln(1 + \frac{\varepsilon\sigma}{L})}{2\alpha} \leq \delta_k \tag{4.6}$$

and choose $\eta > 0$ such that

$$1 - e^{-\lambda r_0} < \eta.$$

Set

$$\theta = \frac{\min_j l_j}{3(\frac{L}{\sigma} + \varepsilon)}, \qquad \beta = \frac{1}{(3 - 2\eta)(1 - \eta)}$$

and

$$s = \frac{\log \beta}{\log \theta}.$$

Since $\min_j l_j \leq \frac{L}{\sigma}$ thus $s < 1$.

We will show that there exists $C > 0$ such that

$$\mu_*(B(\overline{x}, k), r) \leq Cr^s \tag{4.7}$$

for every $r > 0$. Set

$$M = \frac{2L(\frac{L}{\sigma} + \varepsilon)e^{\alpha r_0}}{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}$$

and

$$C = \max\left\{(\theta r_0)^{-s}, \frac{\lambda}{\eta} r_0, M^{\frac{s}{1-s}}\right\}, \tag{4.8}$$

$$r_* = \inf\{\overline{r} > 0 \quad : \mu_*(B(\overline{x}, k), r) \leq Cr^s, \quad \text{for} \quad r > \overline{r}\}.$$

Obviously, condition (4.7) holds for all $r \geq r_0$. Observe that

$$r_* \leq M^{-1/(1-s)}. \tag{4.9}$$

We claim that $r_* = 0$. Suppose, contrary to our claim that $r_* > 0$ and choose $r \in (\frac{r_*}{3(\frac{L}{\sigma}+\varepsilon)}, r_*]$ such that

$$\mu_*(B(\overline{x}, k), r \min_j l_j) > C(r \min_j l_j)^s. \tag{4.10}$$

Define

$$x_0 = \Pi_k(-\overline{t}, \overline{x}), \qquad x_1 = \Pi_k(\overline{t}, \overline{x})$$

where $\overline{t} = r_0(r \min_j l_j)^s$. Further, let

$$B_1 = B\left((\overline{x}, k), \left(\frac{L}{\sigma} + \varepsilon\right) r\right), \quad B_2 = B((x_0, k), r),$$

$$B_3 = B\left((x_1, k), \left(\frac{L}{\sigma} + \varepsilon\right) r\right).$$

Now, let $(y, i) \in B_2$ then by (3.3) and (4.3) we have

$$\|\Pi_k(\bar{t}, y) - \bar{x}\| \leq \frac{L}{\sigma} e^{\alpha \bar{t}} \|y - x_0\| \leq \frac{L}{\sigma} e^{\alpha r_0} r < \left(\frac{L}{\sigma} + \varepsilon\right) r.$$

Therefore

$$B_2 \subset \{(y, i); \quad (\Pi_i(\bar{t}, y), i) \in B_1\}.$$

Using this inclusion and Lemma 4.1 we obtain

$$\mu_*(B_1) \geq e^{-\lambda \bar{t}} \int_{Y \times I} 1_{B_1}(\Pi_i(\bar{t}, y), i) d\mu_*(y, i) \geq e^{-\lambda \bar{t}} \mu_*(B_2) \geq (1 - \eta)\mu_*(B_2). \tag{4.11}$$

Analogously one can show that

$$\mu_*(B_3) \geq (1 - \eta)\mu_*(B_2). \tag{4.12}$$

From (4.5), we have

$$\|x_1 - \bar{x}\| = \|\Pi_k(\bar{t}, \bar{x}) - \bar{x}\| \geq c_{\bar{x},k} \bar{t} \geq \frac{\sigma}{L} e^{-\alpha r_0} c_{\bar{x},k} \bar{t}$$

and

$$\|\bar{x} - x_0\| \geq \frac{\sigma}{L} e^{-\alpha \bar{t}} c_{\bar{x},k} \bar{t} \geq \frac{\sigma}{L} e^{-\alpha r_0} c_{\bar{x},k} \bar{t}. \tag{4.13}$$

Since

$$r < \left(\frac{\sigma c_{\bar{x},k} r_0 (\min_j l_j)^s}{2L\left(\frac{L}{\sigma} + \varepsilon\right) e^{\alpha r_0}}\right)^{1/(1-s)}$$

and $\bar{t} = r_0 (r \min_j l_j)^s$ we obtain

$$\|x_1 - \bar{x}\| > 2\left(\frac{L}{\sigma} + \varepsilon\right) r$$

and

$$\|\bar{x} - x_0\| > 2\left(\frac{L}{\sigma} + \varepsilon\right) r.$$

Thus B_1, B_2, B_3 are mutually disjoint and

$$B_1 \cup B_2 \cup B_3 \subset B\left((\bar{x}, k), 3\left(\frac{L}{\sigma} + \varepsilon\right) r\right).$$

Set $B_4 = B((\bar{x}, k), r \min_j l_j)$. Now we are going to verify that

$$\mu_*(B_2) > (1 - \eta)\mu_*(B_4). \tag{4.14}$$

Let us suppose that, contrary to the above

$$\mu_*(B_2) \le (1-\eta)\mu_*(B_4). \tag{4.15}$$

Since $(\Pi_k(\bar{t}, y), k) \notin B_4$ for $(y, k) \notin B_2$, we have

$$\mu_*(B_4) \le e^{-\lambda\bar{t}} \int_{Y\times I} 1_{B_4}(\Pi_i(\bar{t}, y), i)\,d\mu_*(y, i) + 1 - e^{-\lambda\bar{t}} \le e^{-\lambda\bar{t}}\mu_*(B_2)$$
$$+ 1 - e^{-\lambda\bar{t}} \le \mu_*(B_2) + 1 - e^{-\lambda\bar{t}}.$$

From the last inequality and (4.15) it follows immediately that

$$\mu_*(B_4) \le \frac{1-e^{-\lambda\bar{t}}}{\eta} \le \frac{\lambda\bar{t}}{\eta} = \frac{\lambda r_0}{\eta}(r \min_j l_j)^s.$$

Consequently by the choice of C we obtain

$$\mu_*(B_4) \le C(r \min_j l_j)^s$$

contrary to (4.10).

Further, from (4.11), (4.12) and (4.14) it follows that

$$\mu_*\left(B(\bar{x}, k), 3\left(\frac{L}{\sigma} + \varepsilon\right) r\right) \ge (3-2\eta)\mu_*(B_2) \ge (3-2\eta)(1-\eta)\mu_*(B_4),$$

thus

$$\mu_*(B_4) \le \frac{\mu_*(B(\bar{x}, k), 3(\frac{L}{\sigma} + \varepsilon)r)}{(3-2\eta)(1-\eta)}.$$

By the last inequality, the fact that $3(\frac{L}{\sigma} + \varepsilon)r > r_*$ we have

$$\mu_*(B_4) \le \frac{C\left(3\left(\frac{L}{\sigma} + \varepsilon\right)r\right)^s}{(3-2\eta)(1-\eta)} = \frac{\left(3\left(\frac{L}{\sigma} + \varepsilon\right)\right)^s C(r \min_j l_j)^s}{(\min_j l_j)^s (3-2\eta)(1-\eta)}.$$

Since

$$\left(\frac{3\left(\frac{L}{\sigma} + \varepsilon\right)}{\min_j l_j}\right)^s = (3-2\eta)(1-\eta),$$

thus

$$\mu_*(B_4) \le C(r \min_j l_j)^s$$

which contradicts (4.10). Thus $r_* = 0$ and

$$\mu_*(B(\bar{x}, k), r) \le C r^s \qquad \text{for} \quad r > 0.$$

From the last statement it follows that $\underline{d}\mu_*(\bar{x}, k) \ge s$. Letting $\varepsilon \to 0$ and $\eta \to 0$, we complete the proof in Case I.

Case II. Suppose now that $\alpha < 0$. The proof runs analogously to the proof in Case I and we only point out the main differences of it.

Let $\overline{x} \in Y$ and $k \in I$ be fixed. Choose $\varepsilon > 0$ such that

$$r_0 = \varepsilon \leq \delta_k$$

and set

$$C = \max\left\{(\theta r_0)^{-s}, \frac{\lambda}{\eta} r_0, \left(\frac{2L\left(\frac{L}{\sigma}+\varepsilon\right)}{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}\right)^{s/(1-s)}\right\}$$

in place of (4.6) and (4.8). Define the other constants as in Case I. Then

$$r_* \leq \left(\frac{\sigma c_{\overline{x},k} r_0 (\min_j l_j)^s}{2L\left(\frac{L}{\sigma}+\varepsilon\right)}\right)^{1/(1-s)}.$$

In place of (4.13), we have

$$\|x_1 - \overline{x}\| = \|\Pi_k(\overline{t}, \overline{x}) - \overline{x}\| \geq c_{\overline{x},k}\overline{t} \geq \frac{\sigma}{L} c_{\overline{x},k}\overline{t}$$

and

$$\|\overline{x} - x_0\| \geq \frac{\sigma}{L} e^{-\alpha\overline{t}} c_{\overline{x},k}\overline{t} \geq \frac{\sigma}{L} c_{\overline{x},k}\overline{t}.$$

Argument similar to that of Case I gives that B_1, B_2, B_3 are mutually disjoint. The rest of the proof runs as before. □

5. UPPER BOUND FOR THE GENERALIZED RENYI DIMENSION OF AN INVARIANT MEASURES

The Hausdorff dimension and the generalized Renyi dimension (the concentration dimension) are related by the following two propositions (see ref. 13).

Proposition 5.1. *Let $\mu \in \mathcal{M}_1$ and let $A \in \mathcal{B}(Y)$ be such that $\mu(A) > 0$ Then*

$$dim_H A \geq \underline{dim}_L \mu.$$

Proposition 5.2. *Let $A \subset Y$ be a nonempty compact set. Then*

$$dim_H A = \sup \underline{dim}_L \mu,$$

where the supremum is taken over all $\mu \in \mathcal{M}_1$ such that $supp\mu \subset A$.

To prove the main results of this section we need the following

Lemma 5.1. *Let the numbers $a_i \in [0,1]$ and $b_i \in (0,1)$ for $i \in J$, be given (here J is an arbitrary set of indexes). Let μ be a probability measure. Assume that for some $c > 0$ the Levy concentration function Q_μ satisfies the following condition*

$$Q_\mu(r) \geq \sup_{i\in J} a_i Q_\mu\left(\frac{r}{b_i}\right) \quad \text{for} \quad r \in (0,c). \tag{5.1}$$

Then

$$\overline{dim}_L \mu \leq \inf_{i\in J} \frac{\log a_i}{\log b_i}. \tag{5.2}$$

Proof. The proof can be found in ref. 13. □

Theorem 5.2. *Let the assumptions of Theorem 3.1 hold and let μ_0 be the unique invariant distribution with respect to the operator P given by (3.9). In addition assume that*

$$\sigma = \inf_{x\in Y, i,j\in I} p_{ij}(x) > 0 \tag{5.3}$$

and

$$\gamma = \inf_{x\in Y, s\in S} \overline{p}_s(x) > 0. \tag{5.4}$$

Finally, we assume that

$$\frac{L_q L}{\sigma} < 1. \tag{5.5}$$

Then

$$\overline{dim}_L \mu_0 \leq \frac{\log \sigma\gamma}{\log \frac{LL_q}{\sigma}} \tag{5.6}$$

for $\alpha \leq 0$ and

$$\overline{dim}_L \mu_0 \leq \inf_{M\in(M_0,1)} \frac{\log(\sigma\gamma(1 - M^{\frac{\lambda}{\alpha}}))}{\log \frac{LL_q}{\sigma M}} \tag{5.7}$$

where $M_0 = \frac{L_q L}{\sigma}$ for $\alpha > 0$.

Proof. Let $\overline{x} \in Y$ and $k \in I$ be fixed. By (3.4) there exists $s_0 = s_0(\overline{x})$ (depend on $\overline{x}$) such that

$$\|q_{s_0}(\overline{x}) - q_{s_0}(y)\| \leq L_q \|\overline{x} - y\| \quad \text{for} \quad y \in Y. \tag{5.8}$$

From (3.3), (5.3) and (5.8) we have

$$\|q_{s_0}(\Pi_k(t,x)) - q_{s_0}(\overline{x})\| \leq L_q \|\Pi_k(t,x) - \overline{x}\| \leq \frac{LL_q}{\sigma} e^{\alpha t} \|x - \Pi_k(-t,\overline{x})\|. \tag{5.9}$$

Therefore

$$\begin{aligned}&\{x \in Y : \quad (x,k) \in B\left((\Pi_k(-t,\overline{x}),k), \frac{r\sigma}{LL_q e^{\alpha t}}\right)\\ &\subset \{x \in Y : \quad (q_{s_0}(\Pi_k(t,x)),k) \in B((q_{s_0}(\overline{x}),k),r)\}.\end{aligned}$$

Since μ_0 is invariant, from (3.9) it follows that

$$\begin{aligned}&\mu_0(B((q_{s_0}(\overline{x}),k),r)\\ &\geq \sigma\gamma \int_0^{+\infty} \int_{Y\times I} 1_{B((q_{s_0}(\overline{x}),k),r)}(q_{s_0}(\Pi_k(t,x)),k)\lambda e^{-\lambda t} dt d\mu_0(x,i)\\ &\geq \sigma\gamma \int_0^{+\infty} \mu_0\left(B\left((\Pi_k(-t,\overline{x}),k), \frac{r\sigma}{LL_q e^{\alpha t}}\right)\right)\lambda e^{-\lambda t} dt.\end{aligned} \tag{5.10}$$

This, in turn implies

$$Q_{\mu_0}(r) \geq \sigma\gamma \int_0^{+\infty} Q_{\mu_0}\left(\frac{r\sigma}{LL_q e^{\alpha t}}\right)\lambda e^{-\lambda t} dt. \tag{5.11}$$

Now we consider two cases: $\alpha \leq 0$ and $\alpha > 0$. Suppose first that $\alpha \leq 0$, then

$$Q_{\mu_0}\left(\frac{r\sigma}{LL_q e^{\alpha t}}\right) \geq Q_{\mu_0}\left(\frac{r\sigma}{LL_q}\right) \quad \text{for} \quad t > 0 \quad \text{and} \quad r > 0.$$

Consequently, the function Q_{μ_0} satisfies the inequality

$$Q_{\mu_0}(r) \geq \sigma\gamma Q_{\mu_0}\left(\frac{r\sigma}{LL_q}\right) \quad \text{for} \quad r > 0.$$

From this and Lemma 5.1 we obtain

$$\overline{dim}_L \mu_0 \leq \frac{\log \sigma\gamma}{\log \frac{LL_q}{\sigma}}.$$

Suppose now that $\alpha > 0$. Choose $M < 1$ such that $\frac{LL_q}{\sigma} < M$. Then from (5.11) we obtain

$$Q_{\mu_0}(r) \geq \sigma\gamma \int_0^{\overline{t}} Q_{\mu_0}\left(\frac{r\sigma}{LL_q e^{\alpha t}}\right)\lambda e^{-\lambda t} dt \geq \sigma\gamma Q_{\mu_0}\left(\frac{M\sigma r}{LL_q}\right)(1 - e^{-\lambda\overline{t}}) \tag{5.12}$$

where $\bar{t} = -\frac{\ln M}{\alpha}$. From this and Lemma 5.1 we obtain

$$\overline{dim}_L \mu_0 \leq \frac{\log(\sigma\gamma(1 - M^{\frac{\lambda}{\alpha}}))}{\log \frac{LL_q}{\sigma M}}.$$

□

Define

$$L_0 = \inf_{s \in S} \inf \left\{ \frac{\|q_s(x) - q_s(y)\|}{\|x - y\|}, x \neq y \right\}.$$

Remark 5.1. *Let the assumptions of Theorem 5.1 hold and assume that* $L_0 > 0$. *Then the unique invariant distribution* μ_* *with respect to the semigroup* $\{P^t\}_{t \geq 0}$ *satisfies*

$$\overline{dim}_L \mu_* \leq s$$

where $s = \frac{\log \sigma\gamma}{\log \frac{LL_q}{\sigma}}$ *for* $\alpha \leq 0$ *and* $s = \frac{\log(\sigma\gamma(1-M^{\frac{\lambda}{\alpha}}))}{\log \frac{LL_q}{\sigma M}}$ *for* $\alpha > 0$.

Proof. Let μ_0 be the unique invariant measure with respect to the Markov operator P given by (3.9). From Theorem 5.1 it follows that

$$\overline{dim}_L \mu_0 \leq s.$$

On the other hand, it is known (see ref. 6) that

$$\overline{dim}_L \mu_* \leq \overline{dim}_L \mu_0.$$

□

6. APPLICATIONS

6.1. Example

6.1.1. Learning System

Consider a dynamical system of the form $I = \{1\}$ and $\Pi_1(t, x) = x$ for $t \in \mathbb{R}_+$ and $x \in Y$. Moreover, assume that $p_1(x) = 1$ and $p_{11}(x) = 1$ for $x \in Y$. Then we obtain an iterated function system $(Q, \overline{p}) = (q_1, \ldots, q_K, \overline{p}_1, \ldots, \overline{p}_K)$ with continuous functions $q_s : Y \to Y$, $s \in S = \{1, \ldots, K\}$ and with state dependent probability vector $\overline{p} = (\overline{p}_1, \ldots, \overline{p}_K)$ where $\overline{p}_s : Y \to [0, 1]$ and $\sum_{s=1}^{K} \overline{p}_s(x) = 1$ for $x \in Y$. This system is quite often called a learning system. The system learns because in a new position x_n it uses a new strategy $\overline{p}(x_n)$ for choosing the next step.

A transition operator corresponding to learning system $(Q, \overline{p})$ is given by

$$\overline{P}\mu(A) = \sum_{s \in S} \int_Y 1_A\big(q_s(x)\big)\overline{p}_s(x)\, \mu(dx) \quad \text{for} \quad A \in \mathcal{B}(Y), \quad \mu \in \mathcal{M}_1(Y). \tag{6.1}$$

From Theorem 5.1 we immediately obtain the following result, which belongs to Lasota and Myjak.[13]

Theorem 6.1. *Let $(Q, \overline{p})$ be an iterated function system having an invariant measure $\mu_0 \in \mathcal{M}_1$. Assume that the transformations $q_s : Y \to Y$, $s \in S$ satisfy the Lipschitz condition*

$$\|q_s(x) - q_s(y)\| \leq L_q \|x - y\| \quad \text{for} \quad x, y \in Y$$

with $L_q < 1$ and

$$\gamma = \inf_{x \in Y, s \in S} \overline{p}_s(x) > 0.$$

Then

$$\overline{dim}_L \mu_0 \leq \frac{\log \gamma}{\log L_q}.$$

6.2. Example

6.2.1. Poisson Driven Stochastic Differential Equation

Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. Consider a stochastic differential equation

$$d\xi = a(\xi)dt + b(\xi)dp \quad \text{for} \quad t > 0 \tag{6.2}$$

with the initial condition

$$\xi(0) = \xi_0, \tag{6.3}$$

where $a, b : Y \to Y$ are Lipschitzian functions, Y is a separable Banach space, $\{p(t)\}_{t \geq 0}$ is a Poisson process and the initial condition ξ_0 is a random variable on Ω with values in Y, independent on $\{p(t)\}_{t \geq 0}$.

Let $S = I = \{1\}$ and let $\Pi_1(t, x) = \Pi(t, x)$ be the unique solution of the Cauchy problem

$$\frac{du}{dt} = a(u(t)), \quad u(0) = x. \tag{6.4}$$

Moreover, let $q_1(x) = q(x) = x + b(x)$. Let $x \in Y$, by $\xi(t)_x$ denote the solution of problem (6.2),(6.3) with $x_0 = x$. Then, for every $t \geq 0$ and $f \in C(Y)$ define

$$U^t f(x) = E(f(\xi(t)_x)).$$

Moreover, for every $t \geq 0$ there exists the operator $P^t : \mathcal{M}_s \to \mathcal{M}_s$ satisfying the duality condition

$$\langle f, P^t \mu \rangle = \langle U^t f, \mu \rangle \quad \text{for} \quad f \in B(Y), \quad \mu \in \mathcal{M}_1. \tag{6.5}$$

In many applications we are mostly interested in the values of the solution $\xi(t)$ at the switching points t_n , where $\{t_n\}_{n\geq 0}$ is a sequence of random variables $t_n : \Omega \to \mathbb{R}_+$ with $t_0 = 0$ and such that the increment $\Delta t_n = t_n - t_{n-1}$, $n \in \mathbb{N}$ are independent and have the same density $g(t) = \lambda e^{-\lambda t}$. Set $x_n = \xi(t_n)$ and denote by $\mu_n(A) = \mathbb{P}(x_n \in A)$. It is easy to check that $\mu_n = P^n\mu$, $n \in \mathbb{N}$, where P is the transition operator corresponding to the above stochastic equation and given by

$$P\mu(A) = \int_Y \int_{\mathbb{R}_+} \lambda e^{-\lambda t} 1_A(q(\pi(t,x)))dt\,\mu(dx) \quad A \text{ a Borel subset of } Y. \tag{6.6}$$

From Theorems 5.1 and 5.2 we obtain the following result, due to Myjak and Szarek[17]

Theorem 6.2. *Let Π be the solution of unperturbed systems (6.4). Assume that there exist positive constants α and L_q such that*

$$\|x - y\| \leq \|\Pi(t,x) - \Pi(t,y)\| \leq e^{\alpha t}\|x - y\| \quad \text{for} \quad x, y \in Y, t \geq 0, \tag{6.7}$$

$$\|q(x) - q(y)\| \leq L_q\|x - y\| \tag{6.8}$$

and

$$L_q < \exp\left(-\frac{\alpha}{\lambda}\right). \tag{6.9}$$

Finally we assume that $a(x) \neq 0$ for $x \in Y$. Let μ_* and μ_0 be the invariant distributions with respect to the semigroup P^t given by (6.5) and the operator P given by (6.6) respectively. Then

$$\underline{d}\mu_*(x) \geq 1 \quad for \quad x \in Y \tag{6.10}$$

and

$$\overline{dim}_L\mu_0 \leq \frac{\ln(1 - e^{-1})}{\ln L_q + \frac{\alpha}{\lambda}}. \tag{6.11}$$

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